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Electromagnetic fields in a 3D cavity and in a waveguide with oscillating walls

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Abstract

We consider classical and quantum electromagnetic fields in a three-dimensional (3D) cavity and in a waveguide with oscillating boundaries of the frequency Ω. The photons created by the parametric resonance are distributed in the wave number space around $\Omega/2$ along the axis of the oscillation. When classical waves propagate along the waveguide in the one direction, we observe the amplification of the original waves and another wave generation in the opposite direction by the oscillation of side walls. This can be understood as the classical counterpart of the photon production. In the case of two opposite walls oscillating with the same frequency but with a phase difference, the interferences are shown to occur due to the phase difference in the photon numbers and in the intensity of the generated waves.

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The Casimir effect [1] is a macroscopic manifestation of the change in the zero-point electromagnetic energy due to the walls. The time-varying boundary conditions induce the change of the vacuum states for the quantum electromagnetic fields and the difference between initial and final vacuum states results in the photon production. This dynamical Casimir effect provides the possibility to observe experimentally the vacuum change of quantum fields. This phenomenon has been extensively studied when the one of the walls oscillates [2–5]: For an almost sinusoidal movement of the mirror, using the formalism invented by Moore [6] and developed by Fulling and Davies [7], the quantum energy density has been calculated [8,9]. For a harmonic oscillation of the mirror Méplan and Gignoux have shown that a set of frequencies of the oscillating walls leads to an exponential growth of the energy of a wave [10] and the exponential growth of the number of generated photons can be easily understood from the Floquet's theorem [11]. The scattering approach is used in analyzing the motion induced radiation from a vibrating cavity with partly transmitting mirror(s) [12,13]. For the small oscillation of the walls, the perturbation approach [14] have been developed to calculate the time evolution of the electromagnetic field in the instantaneous basis [15–18].

The aim of this Letter is to examine the photon production in a 3D cavity and to consider the classical electromagnetic fields propagating in a waveguide with oscillating walls. The photons created by the parametric resonance are distributed in the wave number space around the half of the oscillation frequency along the axis of the oscillating motion. We shall show that if we transmit the classical waves into the waveguide with oscillating walls, the waves are amplified and there are generated waves propagating in the opposite direction, which corresponds to the photon production in the quantum theory. When two walls oscillate we find the interference phenomena in the photon numbers and in the intensity of the generated waves.

Assuming that the electric field $\mathbf{E}(\mathbf{r}, t)$ is polarized in the z direction, we may write [6]

$$\begin{aligned} \mathbf{A} &= A(x,y,t)\hat{\mathbf{z}}, \\ \mathbf{E} &= E\hat{\mathbf{z}} = -\frac{\partial A}{\partial t}\hat{\mathbf{z}}, \\ \mathbf{B} &= \frac{\partial A}{\partial y}\hat{\mathbf{x}} - \frac{\partial A}{\partial x}\hat{\mathbf{y}}. \end{aligned} \tag{1}$$

Consider a rectangular cavity with sides $q_x(t)$, L_y and L_z, where the one of the wall oscillates for a time interval $0 < t < T$ with a small amplitude ($\epsilon \ll 1$) according to

$$q_x(t) = L_x(1 + \epsilon \sin \Omega t). \tag{2}$$

In this cavity the field operator can be expanded

$$A = \sum_{\mathbf{n}} [b_{\mathbf{n}}\psi_{\mathbf{n}} + b^{\dagger}_{\mathbf{n}}\psi^{*}_{\mathbf{n}}] \tag{3}$$

using the following instantaneous basis

$$\psi_{\mathbf{n}}(x,y|q_x(t)) = \sum_{\mathbf{k}} Q_{\mathbf{nk}}\varphi_{\mathbf{k}}(x,y,t) \tag{4}$$

where

$$\varphi_{\mathbf{k}}(x,y|q_x(t)) = \frac{2}{\sqrt{q_x L_y L_z}} \sin\frac{\pi k_x x}{q_x} \sin\frac{\pi k_y y}{L_y}, \tag{5}$$

with $k_x, k_y = 1, 2, 3, ...$ From the Maxwell's equations or the wave equation for $\psi_{\mathbf{n}}$ we have

$$\begin{aligned}\ddot{Q}_{\mathbf{nk}} &= -\omega_{\mathbf{k}}^2 Q_{\mathbf{nk}} + 2\epsilon(\pi k_x/L_x)^2 \sin\Omega t Q_{\mathbf{nk}} \\ &\quad +2\epsilon\Omega\cos\Omega t \sum_{\mathbf{j}} g_{\mathbf{kj}}\dot{Q}_{\mathbf{nj}} - \epsilon\Omega^2 \sin\Omega t \sum_{\mathbf{j}} g_{\mathbf{kj}} Q_{\mathbf{nj}} \\ &\quad +O(\epsilon^2)\end{aligned} \tag{6}$$

with

$$g_{\mathbf{jk}} = (-1)^{j_x+k_x} \frac{2 j_x k_x}{k_x^2 - j_x^2} \delta_{j_y k_y} \ (j_x \neq k_x), \tag{7}$$

where $g_{\mathbf{jk}} = 0$ for $j_x = k_x$. Using the perturbation method developed in Ref. [14], the solution can be written as

$$Q_{\mathbf{nk}} = Q_{\mathbf{nk}}^{(0)} + \epsilon Q_{\mathbf{nk}}^{(1)} + \cdots \tag{8}$$

where

$$Q_{\mathbf{nk}}^{(0)}(t) = \frac{e^{-i\omega_{\mathbf{k}} t}}{\sqrt{2\omega_{\mathbf{k}}}} \delta_{\mathbf{nk}} \tag{9}$$

and

$$Q_{\mathbf{nk}}^{(1)}(t) = \sum_{\sigma,s=\pm} w_{\mathbf{k}\sigma,\mathbf{n}-}^{s} \frac{e^{\sigma i\omega_{\mathbf{k}} t}}{\sqrt{2\omega_{\mathbf{k}}}} \int_0^t dt' e^{-i(\sigma\omega_{\mathbf{k}} - s\Omega + \omega_{\mathbf{n}})t'} \tag{10}$$

with

$$w_{\mathbf{k}\sigma,\mathbf{n}\sigma'}^{s} = \sigma \left[\Omega g_{\mathbf{kn}} \sqrt{\frac{\omega_{\mathbf{n}}}{\omega_{\mathbf{k}}}} \left(\frac{s\Omega}{4\omega_{\mathbf{n}}} + \frac{\sigma'}{2} \right) - s\frac{(k_x \pi/L_x)^2}{2\omega_{\mathbf{k}}} \delta_{\mathbf{kn}} \right]. \tag{11}$$

Here we note that the zeroth order solution describes the field operator at the static situation:

$$A = \sum_{\mathbf{n}} [b_{\mathbf{n}} \phi_{\mathbf{n}} + b_{\mathbf{n}}^{\dagger} \phi_{\mathbf{n}}^{*}] \tag{12}$$

where $\phi_{\mathbf{n}} = \frac{1}{\sqrt{2\omega_{\mathbf{n}}}} e^{-i\omega_{\mathbf{n}} t} \varphi(x,y|L_x)$. After some interval T of the oscillation of the wall, the Heisenberg field operator can be written as

$$A = \sum_{\mathbf{n}} [a_{\mathbf{n}} \phi_{\mathbf{n}} + a_{\mathbf{n}}^{\dagger} \phi_{\mathbf{n}}^{*}] \tag{13}$$

where

$$a_{\mathbf{k}} = \sum_{\mathbf{n}} [b_{\mathbf{n}} \alpha_{\mathbf{nk}} + b_{\mathbf{n}}^{\dagger} \beta_{\mathbf{nk}}^{*}], \tag{14}$$

with

$$\sum_{\mathbf{n}}(|\alpha_{\mathbf{nk}}|^2 - |\beta_{\mathbf{nk}}|^2) = 1. \tag{15}$$

The Bogoliubov coefficient $\beta_{\mathbf{nk}}$ can be read from the solution $Q^{(1)}_{\mathbf{nk}}$ to the leading order in ϵ by retaining dominant terms only ($\omega T \gg 1$):

$$\beta_{\mathbf{nk}} = \epsilon T w^{+}_{\mathbf{k}+,\mathbf{n}-}\delta_{\omega_{\mathbf{n}},\Omega-\omega_{\mathbf{k}}}. \tag{16}$$

Hereafter we introduce the bar notation for the wave number vector: $\bar{\mathbf{n}}$ denotes the wave number vector corresponding to $\mathbf{n}$ or in the components, $(\bar{n}_x, \bar{n}_y) = (n_x\pi/L_x, n_y\pi/L_y)$. Noting that the resonance conditions

$$\omega_{\mathbf{n}} = \Omega - \omega_{\mathbf{k}} \text{ and } \bar{n}_y = \bar{k}_y \tag{17}$$

can be explicitly written as [19]

$$\bar{n}_x^2 = \bar{k}_x^2 - 2\Omega\omega_{\mathbf{k}} + \Omega^2, \tag{18}$$

we have the following number distribution in the created photons:

$$\begin{aligned} N_{\mathbf{k}} &= \sum_{\mathbf{n}} |\beta_{\mathbf{nk}}|^2 \\ &= \left(\frac{\epsilon T}{2}\right)^2 \frac{\bar{k}_x^2[\bar{k}_x^2 - 2\Omega\omega_{\mathbf{k}} + \Omega^2]}{\omega_{\mathbf{k}}(\Omega-\omega_{\mathbf{k}})} \end{aligned} \tag{19}$$

This distribution of created photons is anisotropic in the wave number space (see Fig. 1) and the number of created photons is maximal at the nearest neighbor of

$$\bar{k}_x = \frac{\Omega}{2} \text{ and } \bar{k}_y = 0, \tag{20}$$

which come from the conditions $\partial N_{\mathbf{k}}/\partial \bar{k}_x = 0$ and $\partial N_{\mathbf{k}}/\partial \bar{k}_y = 0$. In fact, the second condition in (20) means that $k_y = 1$ because $k_y = 0$ means the vanishing field. Note that the maximum number of photons are created for $\omega_{\max} = \Omega/2$ which is the characteristic condition of the parametric resonance near on the axis of oscillation (with the minimum $\bar{k}_y$.)

Now we extend the results to the case when the left and the right walls oscillate with the frequencies Ω_L and Ω_R respectively:

$$N_{\mathbf{k}} = N^L_{\mathbf{k}} + N^R_{\mathbf{k}} - (-1)^{k_x+n_x} 2\sqrt{N^L_{\mathbf{k}}}\sqrt{N^R_{\mathbf{k}}}\cos\phi\,\delta_{\Omega_L,\Omega_R}. \tag{21}$$

where $N^L_{\mathbf{k}}$ and $N^R_{\mathbf{k}}$ are obtained by replacing Ω in (19) with Ω_L and Ω_R, respectively. Here ϕ is the initial phase difference between two oscillations of the walls and n_x is a positive integer satisfying (18) [19]. When $\Omega_L \neq \Omega_R$, the number of generated photons by the parametric resonance is the sum of the photon numbers generated when the left and the right wall oscillates separately. When $\Omega = \Omega_L = \Omega_R$, there is an additional interference term depending on the mode number of x component and the phase difference ϕ. This is just the interference phenomenon found in the 1D case [20]. It is worth noting that when $\phi = 0$ or $\phi = \pi$ whether the interference is constructive or destructive is determined by x-component mode numbers only. Unlike the 1D case, $n_x + k_x$ does not represent the ratio of

the oscillation frequency Ω to the fundamental mode frequency $\omega_{(1,1)} = \sqrt{(\pi/L_x)^2 + (\pi/L_y)^2}$. But for $L_y \gg L_x$ and $\bar{k}_y \approx 0$, $n_x + k_x$ is an integer close to $\gamma = \Omega/\omega_{(1,1)}$. In this case we have a constructive (destructive) interference when $\phi = \pi$ for $\gamma = 2, 4, ...\gamma = 1, 3, ...$) and when $\phi = 0$ for $\gamma = 1, 3, ...\gamma = 2, 4, ...$). For a general 3D mode, it is possible to have a constructive (destructive) interference when $\phi = \pi$ for an odd (even) γ since the mode frequency depends on not only the x-mode number but also the y-mode number.

We now turn to the classical electromagnetic waves propagating in the positive y-direction in the rectangular cavity with an oscillating wall. When the wall of a cavity is static the right-going wave is described by

$$A(x, y, t) = \sum_{\mathbf{n}} f_{\mathbf{n}} \mathcal{N}_{\mathbf{n}} \cos(\bar{n}_y y - \omega_{\mathbf{n}} t) \sin \bar{n}_x x \tag{22}$$

with the normalization constant $\mathcal{N}_{\mathbf{n}} = \sqrt{2/\omega_{\mathbf{n}} \pi L_x L_z}$. Here the sum over $\mathbf{n}$ denotes the sum over $\bar{n}_x = n_x \pi/L_x$ ($n_x = 1, 2, 3, ...$) and the integration over $\bar{n}_y$, and $f_{\mathbf{n}}$ is a real distribution function of the wave number vector $\bar{n} = \bar{n}_x \hat{\mathbf{x}} + \bar{n}_y \hat{\mathbf{y}}$. There is no boundary along the y-direction and $\bar{n}_y$ can be regarded as a continuum limit of $n_y \pi/L_y$ ($L_y \to \infty$). Introducing the propagating instantaneous basis

$$\varphi_{\mathbf{k}}(x, y | q_x(t)) = \frac{1}{\sqrt{\pi q_x(t) L_z}} \sin \frac{\pi k_x x}{q_x(t)} e^{i \bar{k}_y y}, \tag{23}$$

the classical vector potential at any time can be expanded as

$$A(x, y, t) = \sum_{\mathbf{n}} [f_{\mathbf{n}} \sum_{\mathbf{k}} (Q_{\mathbf{nk}} \varphi_{\mathbf{k}} + \text{H.c.})]. \tag{24}$$

With the initial condition (9) and the static-wall solution with $q_x(t) = L_x$ in (23), the field (24) becomes (22). The time evolution of $Q_{\mathbf{nk}}(t)$ is given by solving the same equation (6) and we have the same solution. Then for the oscillating wall we have the following wave :

$$\begin{aligned} A(x, y, t) = & \sum_{\mathbf{n},\mathbf{k}} f_{\mathbf{n}} \alpha_{\mathbf{nk}} \mathcal{N}_{\mathbf{k}} \cos(\bar{k}_y y - \omega_{\mathbf{k}} t) \sin \bar{k}_x x \\ & + \sum_{\mathbf{n},\mathbf{k}} f_{\mathbf{n}} \beta_{\mathbf{nk}} \mathcal{N}_{\mathbf{k}} \cos(\bar{k}_y y + \omega_{\mathbf{k}} t) \sin \bar{k}_x x]. \end{aligned} \tag{25}$$

where $\beta_{\mathbf{nk}}$ is given by (16) to the leading order in ϵ. Thus we have the left-going wave in the y-direction induced by the x-directional oscillation of the wall, with the amplitude of $\mathbf{k}$ th mode being proportional to $\sum_{\mathbf{n}} f_{\mathbf{n}} \beta_{\mathbf{nk}}$ with $|\sum_{\mathbf{n}} f(\bar{\mathbf{n}}) \beta_{\mathbf{nk}}|^2 = f(\sqrt{\bar{k}_x^2 - 2\Omega\omega_{\mathbf{k}} + \Omega^2}, \bar{k}_y) N_{\mathbf{k}}$, where $f(\bar{\mathbf{n}})$ denotes $f_{\mathbf{n}}$ and $N_{\mathbf{k}}$ is given by Eq. (19). When the two side-walls oscillate, we find the interference phenomena again as in the quantum field case.

Before proceeding on analyzing our results, let us survey other works for the 3D cavity with oscillating walls. It has been suggested that a fantastic amount of photons can be generated in a 3D cavity with one plate being performed periodic instantaneous jumps between two stationary positions [3]. However, this result was obtained by neglecting the terms coupled to other frequency modes in Eq. (6), as pointed out in Ref. [18], and the cases not satisfying Eq. (35) in Ref. [3] which correspond to the condition of parametric resonance [21]

was neglected. In fact, such large number can be obtained from the ultraviolet divergence for the instantaneous jump of the frequency. In Ref. [16,18], the 3D problem has been reduced to a decoupled single parametric oscillator: Using the ansatz $Q_{\mathbf{k}} = \xi_{\mathbf{k}}(\epsilon t)e^{-i\omega_{\mathbf{k}}t} + \eta_{\mathbf{k}}(\epsilon t)e^{i\omega_{\mathbf{k}}t}$ together with the assumption that $\xi_{\mathbf{k}}$ and $\eta_{\mathbf{k}}$ are slowly varying functions of time, and averaging over fast oscillations, the coupling terms [second line in Eq. (6)] have been neglected again in the effective theory. In our case, the coupling terms are also considered and they affect the calculation of the Bogoliubov coefficient $\beta_{\mathbf{nk}}$.

One may wonder if the resonance condition (18) cannot be satisfied due to the discreteness of the frequency in the cavity. As stated [19], small deviation from the resonance condition is admitted and it can be satisfied in the continuum limit ($L_x \ll L_y$). Thus, the propagating wave in the wave guide ($L_y \to \infty$) provides a good experimental situation to observe the generation of the wave or the photon production. The intensity of the generated wave is the order of the intensity of the incident wave multiplied by the number of produced photons in the quantum theory. After a very short time, we may observe an amount of created left-going wave if we prepare the incident right-going wave satisfying the resonance condition (17). For an experimental situation to observe N photons per second ($N = 10$ in the Ref. [13], $N = 600$ in the Ref. [18]), it takes only $1/N$ second to get the same intensity of the generated wave as the incident wave, then the incident right-going wave will be also amplified by the order of $1 + N$ according to (15). This phenomenon can be regarded as the classical counterpart of the photon production in the quantum theory and it is easily observable in the experimental situation.

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FIGURES

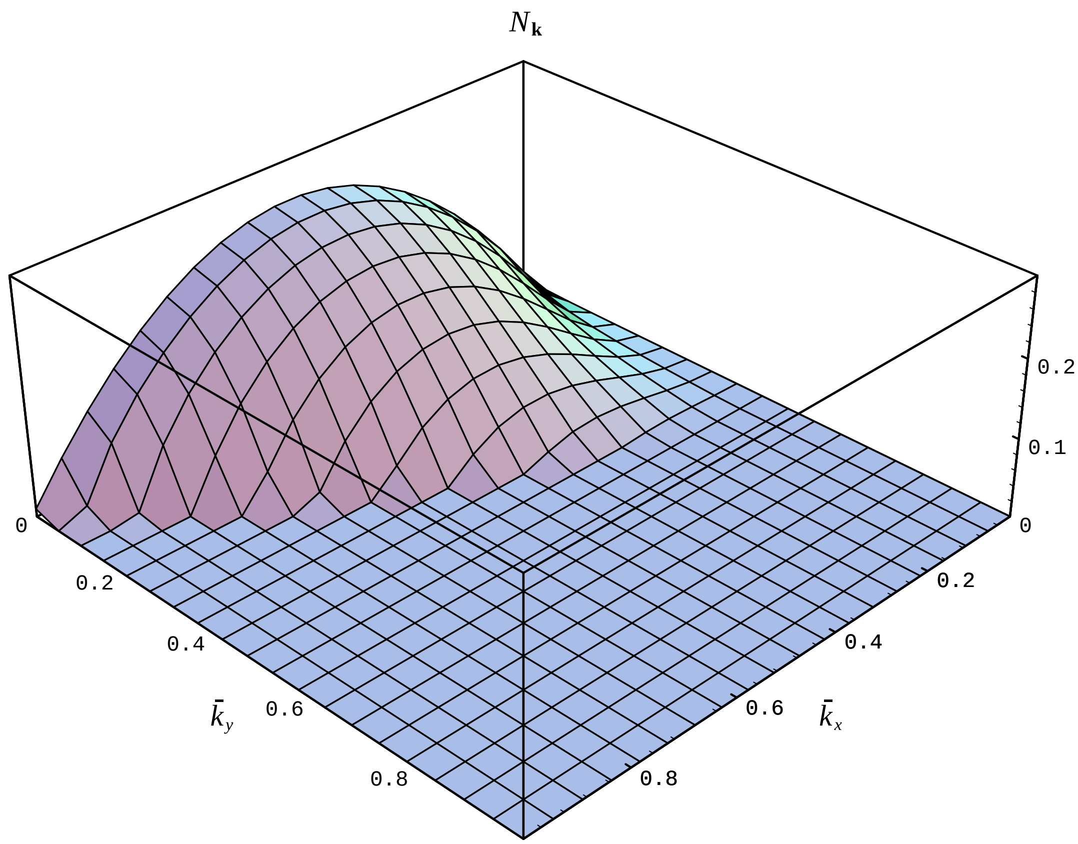

FIG. 1. The distribution of created photons in the rectangular cavity when the side at $x = q_x(t)$ oscillates sinusoidally with a small amplitude. $N_{\mathbf{k}}$ is denoted in units of $(\epsilon \Omega T/2)^2$, and $\bar{k}_x$ and $\bar{k}_y$ are denoted in units of Ω.